

Decomposing Young Quasisymmetric Schur Functions into Dual Immaculate Quasisymmetric Functions

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Big Picture

The Schur functions are one of the most important bases of Sym .
In QSym there are several “Schur like” bases.

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Dual immaculates

- ▶ Introduced by Berg, Bergeron, Saliola, Serrano, Zabrocki in 2014
- ▶ Dual to the Immaculate Basis of NSym
- ▶ Can be defined using a non-commutative version of the Jacobi-Trudi identity
- ▶ Can be defined using Bernstein creation operators.
- ▶ Have a description using tableaux

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Young quasisymmetric Schurs

- ▶ Introduced by Haglund, Luoto, Mason, Mykytiuk, van Willigenburg in 2011/2013
- ▶ Naturally refine the Schur functions in that s_λ is the sum of Young quasi-Schurs indexed by compositions that rearrange to λ .
- ▶ Come from combinatorics of Macdonald polynomials.
- ▶ Have a description using tableaux

Big Picture

- ▶ Using an insertion algorithm, Allen–H–Mason showed that the *dual immaculate basis* expands positively into the *Young quasisymmetric Schur basis*.

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- ▶ Although the reverse expansion is signed, one can ask:
 - ▶ Does it still exhibit meaningful structure?
- ▶ **Answer:** Yes, there seems to be an underlying structure.

The Dual Immaculate Basis

Definition

Let α be a composition of n . A standard **immaculate tableau** is a filling a diagram of shape α with the elements of $\{1, 2, \dots, n\}$ such that

- ▶ Every row is increasing from left to right
- ▶ The leftmost column is increasing from bottom to top.

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Example

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The standard immaculate tableaux of shape $(3, 2)$

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The immaculate tableaux of shape $(3, 2)$ with descents labeled in **blue**.

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where the sum is over all standard immaculate tableaux of shape α .

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Example

2	3	
1	4	5

2	4	
1	3	5

2	5	
1	3	4

3	4	
1	2	5

3	5	
1	2	4

4	5	
1	2	3

$$\mathrm{dI}_{(3,2)} = F_{(1,4)} + F_{(1,2,2)} + F_{(1,3,1)} + F_{(2,3)} + F_{(2,2,1)} + F_{(3,2)}$$

The Young Quasisymmetric Schur Functions

Definition

Let α be a composition of n . A standard **Young composition tableau** of shape α is an immaculate tableau of shape α that satisfies the following **triple rule**: whenever we have the following subarray

$$\begin{array}{|c|c|}\hline b & c \\ \hline\end{array}$$

$\vdots$

$$\begin{array}{|c|}\hline a \\ \hline\end{array}$$

and $a > b$, then $a > c$ (where we take $c = \infty$ if the cell containing c does not appear in the tableau).

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2	3	
1	4	5

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Let T be a standard Young composition tableau, the **Young descent set** of T is

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Example

3	5	
1	2	4

4	5	
1	2	3

The standard Young composition tableaux of shape $(3, 2)$ with descents labeled in **red**.

Immaculate Descents vs Young Descents

Immaculate Descent Set:

$\text{des}_{\uparrow}(T) = \{i \mid i+1 \text{ appears strictly above } i \text{ in } T\}$

Young Descent Set:

$\text{des}_{\leftarrow}(T) = \{i \mid i+1 \text{ appears weakly to the left of } i \text{ in } T\}$

These are not always the same:

2	3	5
1	4	

2	3	5
1	4	

$$\text{des}_{\uparrow}(T) = \{1, 4\}$$

$$\text{des}_{\leftarrow}(T) = \{1, 3\}$$

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Definition

Let α be a composition. The **Young quasisymmetric Schur function** indexed by α is given by

$$\text{YQS}_{\alpha} = \sum_T F_{\text{des}_{\leftarrow}(T)}$$

where the sum is over all standard Young composition tableaux of shape α .

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Example

3	5		
1	2	4	

4	5		
1	2	3	

$$\text{YQS}_{(3,2)} = F_{(2,2,1)} + F_{(3,2)}$$

Dual Immaculate Expansion into Young Quasisymmetric Schurs

Theorem (Allen-H-Mason 2018)

For all compositions α ,

$$dI_\alpha = \sum c_{\alpha,\beta} YQS_\beta$$

where $c_{\alpha,\beta} \geq 0$.

The Decomposition the Other Way

Conjecture

For all compositions α , if

$$\text{YQS}_{\alpha} = \sum b_{\alpha,\beta} d\mathbf{I}_{\beta}$$

then $b_{\alpha,\beta} \in \{-1, 0, 1\}$

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This has been verified (using SageMath) for compositions of n up to $n = 9$

The Decomposition the Other Way

Conjecture

Let λ be a strict partition of n . Then

$$\mathrm{YQS}_\lambda = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \mathrm{dI}_{\sigma(\lambda)}$$

where $\sigma((\lambda_1, \lambda_2, \dots, \lambda_n) = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots, \lambda_{\sigma(n)})$

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Example:

For example,

$$\text{YQS}_{(3,2,1)} = \text{dI}_{(3,2,1)} - \text{dI}_{(3,1,2)} - \text{dI}_{(2,3,1)} + \text{dI}_{(2,1,3)} + \text{dI}_{(1,3,2)} - \text{dI}_{(1,2,3)}$$

This has been verified (using SageMath) for strict partitions of n up to $n = 16$.

Similar Conjectures

The peak algebra is a subalgebra of $\mathbb{Q}\text{Sym}$. Independently Choi, Nam, and Oh (2025) and Searles and Slattery-Holmes (2026) proved the peak analogue of the positive decomposition of dl into YQS.

Similar Conjectures

The peak algebra is a subalgebra of $\mathbb{Q}\text{Sym}$. Independently Choi, Nam, and Oh (2025) and Searles and Slattery-Holmes (2026) proved the peak analogue of the positive decomposition of dl into YQS.

Additionally, Choi, Nam, and Oh (2025) proved that the peak analogue of the strict partition conjecture holds. They also showed that as long as the compositions parts are at most 3, the peak analogue of the general conjecture holds too.

Strategy to Prove the Strict Partition Conjecture

Say we wanted to show that

$$\text{YQS}_{(3,2)} = \text{dI}_{(3,2)} - \text{dI}_{(2,3)}$$

At the level of fundamentals, we have

$$\text{dI}_{(3,2)} - \text{dI}_{(2,3)} = (F_{(1,4)} + F_{1,2,2} + F_{(2,3)} + F_{(1,3,1)} + F_{(2,2,1)} + F_{(3,2)}) - (F_{(1,4)} + F_{1,2,2} + F_{(2,3)} + F_{(1,3,1)})$$

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So we could find a map on immaculate tableaux of shape $(3,2)$ or $(2,3)$ such that

- ▶ The fixed points are the standard Young composition tableaux of shape $(3,2)$
- ▶ The non-fixed points switch signs (i.e swap row lengths).
- ▶ The map preserves descent sets.

Immaculate Descents vs Young Descents Revisited

Proposition

If λ is a strict partition and T is a standard Young composition tableau (hence also an immaculate tableaux) of shape λ , then

$$\text{des}_{\uparrow}(T) = \text{des}_{\leftarrow}(T)$$

Example:

3	5	
1	2	4

3	5	
1	2	4

The map for $\lambda = (3, 2)$

Shape (3, 2)

2	3	
1	4	5

2	4	
1	3	5

3	4	
1	2	5

2	5	
1	3	4

3	5	
1	2	4

4	5	
1	2	3

Shape (2, 3)

2	3	4
1	5	

2	4	5
1	3	

3	4	5
1	2	

2	3	5
1	4	

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1	4	5

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3	4	
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3	5	
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2	3	4
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1	2	

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Intuition behind the Map

As a first step to finding our involution, we are going to encode our immaculate tableaux as a word.

Let T be a standard immaculate tableau. The **word** of T is given by

$$w(T) = w_1 w_2 \dots w_n$$

where w_i is the row of T that i is in.

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$$w \left(\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} \begin{array}{|c|} \hline 5 \\ \hline \end{array} \right) = 12211$$

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We want to swap row lengths. So, we need to change some 1's to 2's and vice versa. But we also need the map to be descent preserving.

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- If we had a word that looked like

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We can't swap the 1 immediately preceding the 2 as this would mean we lose a descent.

- If we had a word that looked like

$\dots 2111 \dots \rightarrow \dots 2112 \dots$ we create a descent!

So we can't swap a 1 if the preceding element is 1 unless we swap that 1 as well. Otherwise we would create a descent.

So, if we swap the red 1, we have to swap all 1's preceding it until we hit a 2.

Sign-reversing, descent-preserving involution

Now suppose that $w = (w_1, w_2, \dots, w_n)$ is the word of a standard immaculate tableau with two rows. We say that a position i is **dominating** if:

- ▶ $w_i = 1$, and
- ▶ starting at position i , and reading cyclically from left to right, the number of 1's is always at least the number of 2's at every point in the traversal.

Dominating Example

We consider cyclic traversals of the word

$$w = 12211$$

We need only consider position 1, 4, and 5.

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So 4 is the unique dominating position.

Dominating Example

Recall that

$$w \left(\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 4 & 5 \\ \hline \end{array} \right) = 122\mathbf{1}1$$

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So, maybe we swap the dominating 1's for 2's?

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Dominating Example

Suppose we apply the same idea to the tableau

$$w \left(\begin{array}{|c|c|} \hline 3 & 5 \\ \hline 1 & 2 & 4 \\ \hline \end{array} \right) = 11212$$

The first dominating 1 is so if we swapped it for a 2 we would get

$$21212$$

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But! that is going to be a problem as the corresponding tableau would be

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which is not immaculate.

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But! that is going to be a problem as the corresponding tableau would be

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which is not immaculate.

Proposition

Let λ be a strict partition with 2 parts and let T be a standard immaculate tableau of shape λ . Then T is a standard Young composition tableau if and only if the first 1 of $w(T)$ is dominating.

Sign-reversing, descent-preserving involution

Suppose w is a word with more 1's than 2's. We define

$$\pi(w) = \begin{cases} w & \text{if the first occurrence of 1 is dominating} \\ w' & \text{otherwise} \end{cases}$$

where w' is obtained from w by replacing each dominating 1 with 2.

We can then extend π to a map on standard immaculate tableaux.

Sign-reversing, descent-preserving involution

T	$w(T)$	$\pi(w(T))$	$\pi(T)$
2 3 1 4 5	12211	12221	2 3 4 1 5
2 4 1 3 5	12121	12122	2 4 5 1 3
3 4 1 2 5	11221	11222	3 4 5 1 2
2 5 1 3 4	12112	12212	2 3 5 1 4
3 5 1 2 4	11212	11212	3 5 1 2 4
4 5 1 2 3	11122	11212	4 5 1 2 3

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2 5 1 3 4	12112	12212	2 3 5 1 4
3 5 1 2 4	11212	11212	3 5 1 2 4
4 5 1 2 3	11122	11212	4 5 1 2 3

Sign-reversing, descent-preserving involution

So far, we know how to map when there are more 1's than 2's. What if it is the other way around? Basically, you look at the 2's and instead considering them as dominating, you look at *reverse-dominating*. This is done by reading backwards cyclically using the same idea of dominating (always making sure there are at least as many 2's as 1's).

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Example:

Say the word was 12221. You can verify that the 2 in position 4 is reverse-dominating.

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Say the word was 12221. You can verify that the 2 in position 4 is reverse-dominating.

Proposition

If position i is dominating in T and $\pi(T) \neq T$, then position i is reverse dominating in $\pi(T)$.

Sign-reversing, descent-preserving involution

Let $\lambda = (\lambda_1, \lambda_2)$ be a strict partition. Let S be the set of standard immaculate tableaux of shape (λ_1, λ_2) or (λ_2, λ_1) . Define

$\pi : S \rightarrow S$ given by

$$\pi(T) = \begin{cases} T & \text{if the first 1 of } w(T) \text{ is dominating} \\ T' & \text{if } \text{shape}(T) = \lambda \text{ and the first 1 of } w(T) \text{ is not dominating} \\ T'' & \text{otherwise} \end{cases}$$

where T' is obtained from T by changing all dominating 1's to 2's and T'' is obtained from T by changing all reverse dominating 2's to 1's.

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Theorem

The map π has the following properties.

- ▶ $\pi(T) = T$ if and only if T is standard Young composition tableau of shape λ
- ▶ $\text{des}_\uparrow(T) = \text{des}_\uparrow(\pi(T))$
- ▶ If $\pi(T) \neq T$, then $\text{shape}(T) \neq \text{shape}(\pi(T))$

Strict Partition Conjecture for 2 Rows

Theorem

Suppose that $\lambda_1 > \lambda_2$. Then

$$\text{YQS}_{(\lambda_1, \lambda_2)} = \text{dI}_{(\lambda_1, \lambda_2)} - \text{dI}_{(\lambda_2, \lambda_1)}$$

Decomposing YQS into dl when we have 2 Rows

For a given partition λ , it is known that

$$s_\lambda = \sum_{\alpha} YQS_{\alpha}$$

where the sum is over all compositions α that rearrange to λ . Moreover, it is known how to decompose a Schur function into the dual immaculate functions. Consequently, we get the following.

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Theorem

For any positive integers α_1 and α_2

$$\text{YQS}_{(\alpha_1, \alpha_2)} = \begin{cases} \text{dI}_{(\alpha_1, \alpha_2)} - \text{dI}_{(\alpha_2, \alpha_1)} & \text{if } \alpha_1 > \alpha_2 \\ \text{dI}_{(\alpha_1, \alpha_2)} - \text{dI}_{(\alpha_1-1, \alpha_2+1)} & \text{if } \alpha_1 \leq \alpha_2 \end{cases}$$

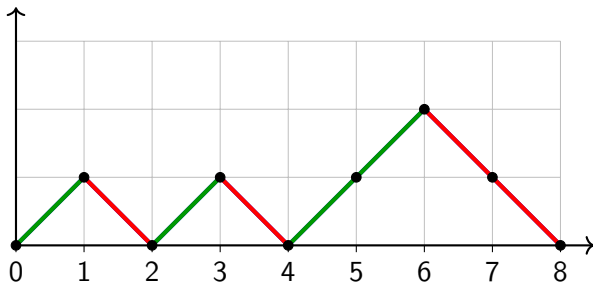
Other Consequences

Let $\lambda = (n + 1, n)$, and let T be a Young composition tableau. Then the reading word $w(T)$ contains exactly one more 1 than 2. Since T is a Young composition tableau, the first occurrence of 1 in $w(T)$ must be dominating. It follows that removing this initial 1 yields a word corresponding to a Dyck path of semilength $n - 1$.

Example:

3	5	8	9	
1	2	4	6	7

 $\rightarrow 112121122 \rightarrow \cancel{1}12121122$



Other Consequences

Theorem

The number of standard Young composition tableaux of shape $(n + 1, n)$ with $|\text{des}(T)| = k$ is the Narayana number $N(n - 1, k)$. Consequently, the number of standard Young composition tableaux of shape $(n + 1, n)$ is $(n - 1)^{\text{th}}$ Catalan number.

Future Work

How do we extend this to shapes with more than 2 rows?

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For each i , we define π_i in a similar way as π . In this case, π_i considers swapping i 's and $i + 1$'s.

Why isn't this just immediate? Well, the real issue is that it isn't clear given a tableau, which π_i we should apply. When there is only two rows, you only have one choice.

Thank You!